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A New Form of Artificial Viscosity for Elastic Solids*

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A form of artificial viscosity, which is second-order and quadratic, is developed for elastic solids. This new form and the standard linear artificial viscosity are compared. It is observed that the artificial viscosity works best when applied only to expanding zones.

INTRODUCTION

Von Neuman and Richtmyer [1] proposed a technique, known as artificial viscosity, for damping numerical noise when simulating gas dynamics on a digital computer. The technique is frequently called the "q" technique because an artificial pressure (designated by the letter q) is calculated and added to the pressure for every zone in the problem. The following equation describes this q:

$$\begin{aligned} q &= \rho(c\Delta x)^2 \,|\, \Delta u/\Delta x \,|^2 && \text{for compressing zones,} \\ &= 0 && \text{for expanding zones,} \end{aligned}$$

where ρ is density, u is velocity, Δx is zone width, and c is a dimensionless constant.

Unfortunately, this form of q does not damp numerical noise when calculating weak shocks in elastic solids. If a q is to have a noticeable damping effect, its value must be significant compared to the value of p, the pressure. A good measure of the damping ability of the q is the ratio of q to p. For weak shocks we also know that

$$\Delta u = \Delta p/\rho c_s,$$

where c_s is the sound speed, and is approximately constant.

For the elastic solid we have

$$\left.\frac{q}{p}\right)_{\mathrm{ES}} = \frac{c^2}{\rho c_s^2}\frac{\Delta p}{p}\Delta p.$$

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A similar calculation for a γ-law gas ($c_s^2 = \gamma p/\rho$) yields

$$\left.\frac{q}{p}\right)_\gamma = \frac{c^2}{\gamma}\left(\frac{\Delta p}{p}\right)^2.$$

Thus for the γ-law gas, the ratio of q to p is determined by the relative size of the change in pressure and the pressure itself. If the change in pressure is large relative to the pressure, the damping will be large; if the change in pressure is small relative to the pressure, the damping will be small. However, for the elastic solid, q/p can vanish when Δp is small even if $\Delta p/p$ is not small. But, if $\Delta p/p$ is not small, significant damping is needed.

A linear q is usually used for damping numerical oscillations resulting from weak shocks in elastic solids. The linear q is given as

$$q_L = \rho(c\Delta x)\, c_s(\Delta u/\Delta x).$$

In the limit of weak shocks we obtain

$$q_L = c\Delta p.$$

And, since this is independent of the equation of state, we obtain

$$q_L/p = c(\Delta p/p),$$

which holds for elastic solids and γ-law gases. The linear q has the disadvantage of being weakly sensitive to the presence of shocks and it introduces an error that is first order in (Δx). This has been the most frequently used q to damp numerical oscillations resulting from weak shocks in elastic solids. Unfortunately, although the noise is damped, the solution may be dominated by the dissipation introduced by the q rather than by the inviscid physics which the model is intended to simulate. Viecelli [2] has discussed this problem and points out that the difficulty can be circumvented only by using unreasonably small zone size. To minimize the error of dissipating too much kinetic energy and of converting it into potential energy, one would like to have a quadratic q (and second-order accurate in Δx) that would work for elastic solids. Consider the following form for q

$$q_{ES} = (c\Delta x)^2(\Delta p/\Delta x)^2(1/\bar{p}),$$

where $\bar{p}$ is the average of the pressure ahead of the shock and the pressure behind the shock. Of course, $\bar{p}$ is an unknown, but it can be approximated by using the third Rankine–Hugoniot equation:

$$\Delta E = \bar{p}\,\Delta(1/\rho), \qquad \text{i.e.,} \qquad \bar{p} = \Delta E/\Delta(1/\rho),$$

where E is the energy per unit mass. Note that, away from shocks, $\bar{p}$ reduces to p. It should also be noted that q_{ES} is equal to q_L multiplied by $(c\Delta x)(\Delta p/\Delta x)(1/\bar{p})$ in the limit of weak shocks.

We obtain the following equation for q_{ES}

$$q_{ES} = (c\,\Delta x)^2 \left| \frac{\Delta p}{\Delta x} \frac{\Delta(1/\rho)}{\Delta x} \right| \left| \frac{\Delta p}{\Delta x} \right| \left| \frac{\Delta x}{\Delta E} \right|.$$

We can replace $\Delta p\, \Delta\, 1/\rho$ by $(\Delta u)^2$ because of the second Rankine–Hugoniot equation. We can also replace $\Delta p/\Delta x$ by $(1/\dot{\xi})(\Delta p/\Delta t)$ and $\Delta E/\Delta x$ by $(1/\dot{\xi})(\Delta E/\Delta t)$, where $\dot{\xi}$ is the signal speed, because of the characteristic nature of wave equations. The equation for q_{ES} then becomes

$$q_{ES} = c^2 \,|\, \Delta u \,|_x^2 \left| \frac{\Delta p}{\Delta E} \right|_t,$$

where the subscript x refers to spatial differences across a zone and t refers to temporal differences within a zone. This form of q_{ES} will not encounter intrinsic difficulties at material interfaces or at boundaries. An important point is that ΔE_t must be purely the contribution resulting from $(p + q)\,dV$ work, that is, it must not include a contribution from the elastic energy change. Of course, if ΔE_t equal zero, the calculation must be skipped and the q_{ES} set equal to zero. One can also consider leaving the q turned on for expanding zones in addition to or in place of compressing zones. The sign for a q is negative when turned on for expanding zones.

Computer Calculations

Numerous simple calculations were run to test this new form of q. The standard quadratic form of Von Neumann and Richtmyer is always used in addition q_{ES} or q_L. Its influence is important for strong shock waves but becomes insignificant for waves that are weak or that become weak. The first series of calculations consisted of a 10-cm-radius sphere of high explosive placed in the center of a larger sphere (radii varying from 100 cm to 1000 cm) of iron using the UKO [3] computer program and its equations of state of these materials.

With the iron radius at 100 cm, a zone size study was performed in which either q_{ES} or q_L were turned for compressing or expanding zones or for both. The high explosive is completely burned at $t = 0$ and after 175 μsec a weak shock has arrived at the $R = 90$ cm position. The peak pressure in the shock at this time is plotted in Fig. 1, as a function of relative zone size for several different q formu-

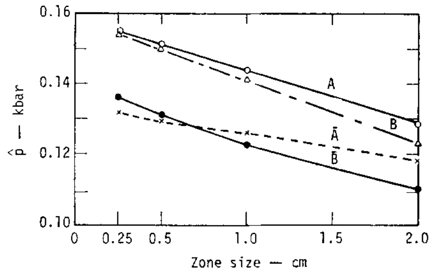

FIG. 1. Peak pressure vs zone size. Curve A is $q_L = -0.1\, \rho\, C_s \mid \Delta U \mid$ for expanding zones only; Curve B is $q_{ES} = -1.0 \mid \Delta P \mid^2/\bar{P}$ for expanding zones only; Curve $\bar{A}$ is $q_L = \pm\, 0.05\, \rho\, C_s \mid \Delta U \mid$ "on" for expanding and compressing zones; and Curve $\bar{B}$ is $q_{ES} = \pm\, 0.25 \mid \Delta P \mid^2/\bar{P}$ "on" for expanding and compressing zones.

lations. The different coefficients were chosen so that all sets of calculations possessed approximately the same degree of damping or "smoothness."

If only the $\bar{A}$ curve has been plotted, it would be easy to obtain an extrapolated value of $\hat{p}$ that would be too small by a significant amount. This is the effect Viecelli discusses. The $\bar{A}$ and $\bar{B}$ curves indicate that a convergence problem exists for the q being "on" for compressing zones for elastic solids. That is, the solutions do not appear to be converging to the same result. Although the curve is not plotted on Fig. 1, a similar q_L calculation was run with the q "on" only for compressing zones. Its plot fell essentially on top of the $\bar{A}$ curve.

The functional behavior of the A and B curves is much more satisfactory than those of $\bar{A}$ and $\bar{B}$. From them, we observe that the zero zone size extrapolated value of $\hat{p}$ is approximately 0.158 kbar. One might anticipate that the solution for $\hat{p}$ of the $\bar{A}$ and $\bar{B}$ curves would also converge to approximately 0.158 kbar if an unlimited number of zones were available. In these sets of calculations, it has been observed the q_{ES} can indeed provide damping for weak shocks, and it has also been observed that the damping for elastic solids should concentrate on the expanding zones.

With the outer radius of the iron increased by a factor of 10 (to 1000 cm), a weak shock arrives at $R \simeq 880$ cm at time 1925 μsec. With $\Delta x = 2.0$ cm the following results for peak pressures ($\hat{p}$) were obtained in two calculations.

$$\hat{p} = 0.055 \text{ kb} \qquad q = \pm 0.1\, \rho\, c_s \mid \Delta u \mid \quad \text{(compression and expansion; linear terms only)} \tag{1}$$

$$\hat{p} = 0.070 \text{ kb} \qquad q = -0.3 \mid \Delta p \mid^2/\bar{p} - 0.07\, \rho\, C_s \mid \Delta U \mid \quad \text{(expansion only; quadratic and linear terms)} \tag{2}$$

Note that the second calculation used both q_{ES} and q_L but that the coefficient for q_L is very small compared to that for q_{ES}. The velocity showed a similar difference at the peak of the weak shock. This of course means an even larger difference in the peak values of kinetic energy. The ratio of the two $\hat{p}$ values is 0.786 for the two different forms of q. At 175 μsec ($R = 90$-cm shock position) the values of $\hat{p}$ had been 0.115 kbar and 0.126 kbar, giving a ratio of 0.913 when the signal has traveled the same distance as was used in the zone size study of Fig. 1. This simply indicates that for longer distances of signal travel, the importance of dissipation increases.

The previous examples apparently indicate some techniques available for qualitative improvements in solution behavior for a type of calculation that is frequently of practical interest. A new second-order quadratic q has been developed such that its damping properties are effective for weak shocks in elastic materials. Furthermore, the use of significant (linear or quadratic) q's for elastic waves should be restricted to expanding zones to reduce nonphysical dissipation and to obtain better convergence properties as the zone size approaches zero.

We will now apply these techniques to a problem with an analytic solution to obtain a more conclusive measure of their utility. A pressure is instantaneously applied at the surface of an elastic planar material and the pressure then decays exponentially in time. The pressure pulse should propagate at the sound speed through the material. For the particular calculation chosen here the elastic constants are: $\lambda = 0.5$ Mbar, $\mu = 0.0$, and $p_0 = 0.001$ Mbar with an e-folding time of 1 msec (λ and μ are Lamé's parameters). Even though the signal is weak, the wave front is discontinuous. Hence, this problem presents a real test of a technique's ability to damp oscillations and still retain accuracy. In the earlier problems (spherical high explosive inside spherical iron) the signal was spread initially by the classical q of von Neumann and Richtmyer so that the new q technique for elastic waves never saw a discontinuous signal. In those problems, the new q handled the transition from strong shocks to weak shocks quite well, maintaining a smooth signal while introducing less dissipation.

Figures 2, 3, and 4 show the degree to which noise is damped as the q is "turned on" for calculations of the analytic solution problem after the pressure has e-folded four times. Figure 2 has no q affecting the noise; in Fig. 3 a small amount of the new quadratic elastic q has damped most of the noise; and in Fig. 4 a very small linear q has been added for additional smoothing. The analytic solution is also shown in each figure and it is obvious that the q has dissipated the signal significantly. Because of the q's zone-size dependence, one would expect noticeably less dissipation as the zone size is reduced. In Fig. 5, the results are presented in which the calculation has been done in the same way as for Fig. 4, except that the zone size has been reduced by a factor of four. Clearly, there has been less dissipation, but a small overshoot is now apparent (approximately 3 %). This can be compared to

Fig. 6, which gives a finely zoned calculation using a more typical form of linear q that is "on" for compressing zones. The linear-compressive q results in an undershoot of about 17%. Thus, the error at the peak is considerably smaller with the new q. Of greater importance than the size of overshoot or undershoot is the amount of dissipation which eats away at the kinetic energy carried by the wave. The kinetic energy carried by the wave is more accurately measured by the *area* under the pressure *vs* position curve. By this criterion, the improvement resulting from the new q technique is even more significant. The longer the wave is propagated, the greater will be the relative improvement resulting from the new q. In fact, it would also have been possible to examine the wave after fewer pressure

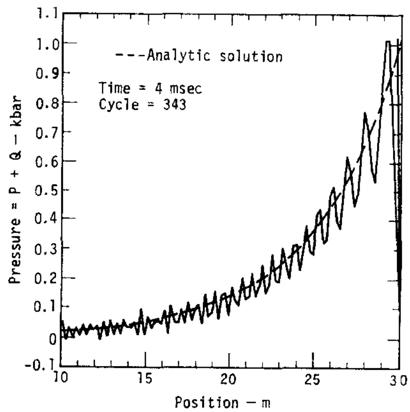

FIG. 2. Pressure vs radius for 100 zones: no q.

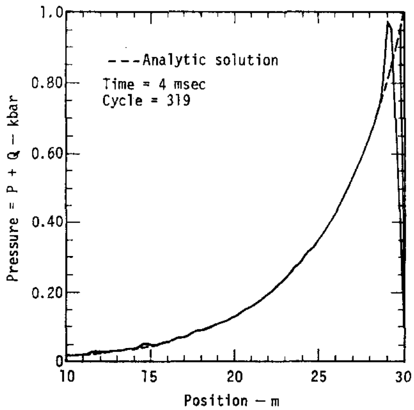

FIG. 3. Pressure vs radius for 100 zones: $q_{ES} = -0.5\,[(\Delta p)^2/\bar{p}]$ (quadratic term only) (expansion zones only).

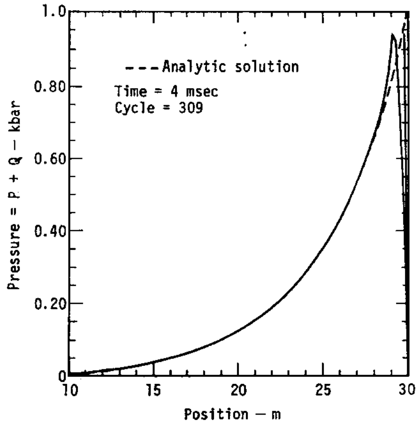

FIG. 4. Pressure vs radius for 100 zones: $q = q_{ES} - 0.05\,\rho c_s\,|\Delta u|$ (expansion zones only).

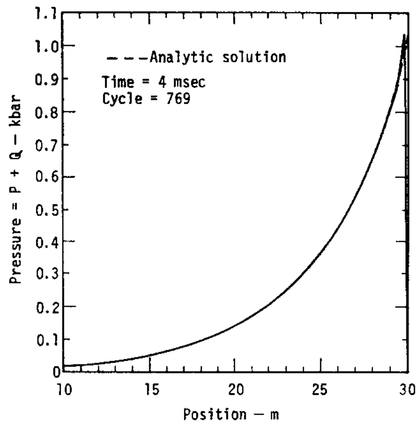

FIG. 5. Pressure vs radius for 400 zones (finely zoned): $q = q_{ES} - 0.05\,\rho c_s\,|\Delta u|$.

e-folds and to have observed that the overshoot and undershoot were of comparable sizes; however the new q would have resulted in considerably less dissipation as indicated by the area under the pressure–position curve. Had we chosen a steeply rising (but not instantaneously rising) pressure pulse with a smooth peak, no significant overshoot would have developed with the new q. This is mentioned to emphasize again that the particular analytic example presented here poses a very severe test of the improvement obtainable by the new technique.

By ensuring that Δt was less than 0.9 the Courant value, stability was achieved in the above calculations. The coefficients chosen for the q's naturally reflect the

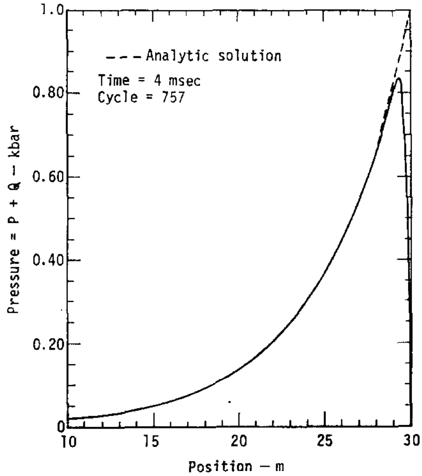

FIG. 6. Pressure vs radius for 400 zones (finely zoned): $q = \pm 0.1\, \rho c_s \mid \Delta u \mid$ (symmetric for expansion and compression zones).

bias of the author's experience, particularly the concentration on spherically diverging waves. The q is formulated in terms of pressure gradients because this is the simplest approach, but the stress gradients might be more appropriate.

DISCUSSION

Von Neumann and Richtmyer [1] showed that the jump conditions across the shock were satisfied by the introduction of artificial viscosity for calculations of ideal gases: That q can be considered to account for irreversible thermodynamics. In the elastic problems described in this paper, better results are obtained by applying a new form of the q (and the old linear form) to expanding zones. For expanding zones, gradients are usually not steep and dissipation is not excessive. For elastic behavior, there are no irreversible thermodynamic phenomena to be considered. Thus it is best to avoid adding artificial viscosity for compressing-zones, since such an addition may result in significant undesirable dissipation in the presence of strong gradients.

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